

On Some Topological Concepts of TVS-Cone Metric Spaces and Fixed Point Theory Remarks

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Abstract

We prove that every TVS-cone metric space (i.e a cone metric space over a locally convex topological vector space E) is first countable paracompact topological space and by using Du's results in " [A note on cone metric fixed point theory and its equivalence, *Nonlinear Analysis*, 72(5), 2259-2261 (2010)]", we conclude that every TVS-cone metric space is topologically isomorphic to a topological metric space. We also show how to construct comparable metric topologies to TVS-cone metric topologies by using the system of seminorms generating the topology of the locally convex topological vector space E . When E is a Banach space these metric topologies turn to be equivalent to the original TVS-cone metric topologies. Even though, we remark that there are still some fixed point theorems to deal nontrivially with them in TVS-cone metric spaces. The nonlinear scalarization is used also to prove some fixed point theorems with nonlinear contractive conditions.

1 Introduction and Preliminaries

The cone metric space fixed point theory was initiated in 2007, by Huang and Zhang [8]. Afterwards, many authors started to generalize fixed point theorems from metric spaces to cone metric spaces (CMS) (see [10],[11],[12],[13],[9],[2],[3],[18]). Through those generalizations, normal, sequentially regular and strongly minihedral cones were crucial. However, some results have been carried to CMS without any extra assumptions on the cone. Some other authors cared about the topological structures in CMS ([18] and [15]). It was proved that CMS are first countable paracompact topological spaces provided that the cone must not have empty interior.

Recently, Du [7] gave the definition of a more generalized cone metric space, namely topological vector space-cone metric space (TVS-CMS) and showed that Banach contraction principle in usual metric spaces and in TVS-CMS are equivalent. Actually, he concluded that certain contractive type conditions can be transformed into equivalent ones in the correspondent usual metric space by applying the nonlinear scalarization. The fixed point theorems (for example

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Kannan's type and Chatterjea's type) containing such kinds of conditions have been generalized before to CMS without any normality or regularity assumption on the cone under consideration.

In this manuscript, we prove that every TVS-CMS is a first countable paracompact topological space without the need of normality assumption (see [15]). The proofs based on that every TVS-CMS is topologically isomorphic to a usual metric space. The key in defining these metric equivalent was the nonlinear scalarization function via some interior point of the cone [7]. However, we also describe how to construct such comparable metric topologies to the TVS-cone metric topologies by using the system of the seminorms generating the topology of the locally convex topological space E . Actually, when E is Banach we arrive at an equivalent topology. Even though, we remark that there are still some fixed point theorems that can not be proved trivially in TVS-CMS by making use of the equivalent metric metric topology. In the trivialization direction, we showed also how to transform some nonlinear contractive type conditions from TVS-CMS to usual metric spaces.

Throughout this paper, (E, S) stands for real Hausdorff locally convex topological vector space (t.v.s.) with S its generating system of seminorms. A non-empty subset P of E is called cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. The cone P will be assumed to be closed with nonempty interior as well, that is, P will be a convex pointed cone with nonempty interior. The positive cones in $C[0, 1]$ and $L_\infty[0, 1]$ have nonempty interiors. However, the positive cone of $L_p[0, 1]$, $0 \leq p < \infty$ have empty interiors [14]. Moreover, it is known that a real locally convex topological vector space has a continuous norm if and only if it contains a closed pointed convex cone with nonempty interior [19]. There are many examples of Fréchet Köthe sequence spaces which admit continuous norms and hence contain closed convex pointed cones with nonempty interiors. For a given cone P , one can define a partial ordering (denoted by \leq : or \leq_P) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$ while $x \ll y$ will show $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P . Continuity of the algebraic operations in a topological vector space and the properties of the cone imply the relations:

$$\text{int}P + \text{int}P \subseteq \text{int}P \text{ and } \lambda \text{int}P \subseteq \text{int}P \ (\lambda > 0).$$

We appeal to these inclusions in the following.

Definition 1. [14] A cone P of a topological vector space (X, τ) is said to be normal whenever τ has a local base of zero consisting of P -full sets, where a subset A is said to be P -full if $\{a \in E : x \leq a \leq y\} \subset A$ for all $x, y \in A$.

Theorem 2. [14] (a) A cone P of a topological vector space (X, τ) is normal whenever $\{x_\alpha\}$ and $\{y_\alpha\}$, $\alpha \in \Delta$ are two nets in X with $0 \leq x_\alpha \leq y_\alpha$ for each $\alpha \in \Delta$ and $y_\alpha \rightarrow 0$, then $x_\alpha \rightarrow 0$.

(b) The cone of an ordered locally convex space (X, τ) is normal if and only if τ is generated by a family of monotone τ -continuous seminorms. Where

a seminorm q on X is called monotone if $q(x) \leq q(y)$ for all $x, y \in X$ with $0 \leq x \leq y$.

In particular, if P is a cone of a real Banach space E , then it is called *normal* if there is a number $K \geq 1$ such that for all $x, y \in E$: $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The least positive integer K , satisfying this inequality, is called the normal constant of P . Also, P is said to be *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n \geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \dots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$. For more details about cones in locally convex topological vector spaces we may refer the reader to [14].

Definition 3. [5] A cone P of a locally convex topological vector space (E, S) is said to be (sequentially) *regular* if every sequence in E which is increasing and bounded above must be convergent in (E, S) . That is, if $\{a_n\}$ is a sequence in E such that $a_1 \leq a_2 \leq \dots \leq a_n \leq \dots \leq b$ for some $b \in E$, then there is $a \in E$ such that $q(a_n - a) \rightarrow 0$ as $n \rightarrow \infty$, for all $q \in S$. Equivalently, if every sequence in E which is decreasing and bounded below is convergent.

It is well-known that if E is a real Banach space then every (sequentially) regular cone is normal.

Definition 4. [5] A cone P of a locally convex topological vector space (E, S) is said to be *strongly minihedral* if and only if any subset of E which is bounded above must have a least upper bound. Equivalently, if any subset of E which is bounded below must have a greatest lower bound.

Definition 5. (See [4], [6], [7]) For $e \in \text{int}P$, the nonlinear scalarization function $\xi_e : E \rightarrow \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{t \in \mathbb{R} : y \in te - P\}, \text{ for all } y \in E.$$

Lemma 6. (See [4], [6], [7]) For each $t \in \mathbb{R}$ and $y \in E$, the following are satisfied:

- (i) $\xi_e(y) \leq t \Leftrightarrow y \in te - P$,
- (ii) $\xi_e(y) > t \Leftrightarrow y \notin te - P$,
- (iii) $\xi_e(y) \geq t \Leftrightarrow y \notin te - \text{int}P$,
- (iv) $\xi_e(y) < t \Leftrightarrow y \in te - \text{int}P$,
- (v) $\xi_e(y)$ is positively homogeneous and continuous on E ,
- (vi) if $y_1 \in y_2 + P$, then $\xi_e(y_2) \leq \xi_e(y_1)$,
- (vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$, for all $y_1, y_2 \in E$.
- (viii) if $y_1 \in y_2 + \text{int}(P)$, then $\xi_e(y_2) < \xi_e(y_1)$

Lemma 6 (vi) and Lemma 6 (viii) mean that the nonlinear scalarization function ξ_e is monotone and strictly monotone. However, it is not strongly monotone (see [4]).

Definition 7. Let X be a non-empty set. and E as usual a Hausdorff locally convex topological space. Suppose a vector-valued function $p : X \times X \rightarrow E$ satisfies:

- (M1) $0 \leq p(x, y)$ for all $x, y \in X$,
- (M2) $p(x, y) = 0$ if and only if $x = y$,
- (M3) $p(x, y) = p(y, x)$ for all $x, y \in X$
- (M4) $p(x, y) \leq p(x, z) + p(z, y)$, for all $x, y, z \in X$.

Then, p is called TVS-cone metric on X and the pair (X, p) is called a TVS-cone metric space (in short, TVS-CMS).

Note that in [8], the authors considered E as a real Banach space in the definition of TVS-CMS. Thus, a cone metric space (in short, CMS) in the sense of Huang and Zhang [8] is a special case of TVS-CMS.

Lemma 8. (See [7]) Let (X, p) be a TVS-CMS. Then, $d_p : X \times X \rightarrow [0, \infty)$ defined by $d_p = \phi_e \circ p$ is a metric.

Remark 9. Since a cone metric space (X, d) in the sense of Huang and Zhang [8], is a special case of TVS-CMS, then $d_p : X \times X \rightarrow [0, \infty)$ defined by $d_p = \xi_e \circ d$ is also a metric.

Definition 10. (See [7]) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X .

- (i) $\{x_n\}_{n=1}^{\infty}$ TVS-cone converges to $x \in X$ whenever for every $0 < c \in E$, there is a natural number M such that $p(x_n, x) < c$ for all $n \geq M$ and denoted by cone $\lim_{n \rightarrow \infty} x_n = x$ (or $x_n \xrightarrow{\text{cone}} x$ as $n \rightarrow \infty$),
- (ii) $\{x_n\}_{n=1}^{\infty}$ TVS-cone Cauchy sequence in (X, p) whenever for every $0 < c \in E$, there is a natural number M such that $p(x_n, x_m) < c$ for all $n, m \geq M$,
- (iii) (X, p) is TVS-cone complete if every sequence TVS-cone Cauchy sequence in X is a TVS-cone convergent.

Lemma 11. (See [7]) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X . Set $d_p = \xi_e \circ p$. Then the following statements hold:

- (i) If $\{x_n\}_{n=1}^{\infty}$ converges to x in TVS-CMS (X, p) , then $d_p(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$,
- (ii) If $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in TVS-CMS (X, p) , then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence (in usual sense) in (X, d_p) ,

(iii) If (X, p) is complete TVS-CMS, then (X, d_p) is a complete metric space.

Remark 12. Note that the implications in (i) and (ii) of Lemma 11 are also conversely true. Regarding (i) we prove that if $x_n \rightarrow x$ in (X, d_p) , then $x_n \rightarrow x$ in (X, p) . To this end, let $c \gg 0$ be given, and find $q \in S$ and $\delta > 0$ such that $q(b) < \delta$ implies that $b \ll c$. This is possible since we can find a symmetric neighborhood of c , $c + \{b : q(b) < \delta\} \subset \text{int}P$. Since $\frac{c}{n} \rightarrow 0$ in (E, S) , where $c \gg 0$, find $\epsilon = \frac{1}{n_0}$ such that $\epsilon q(c) = q(\epsilon c) < \delta$ and hence $\epsilon c \ll c$. Now, find n_0 such that $d_p(x_n, x) = \xi_{\epsilon} \circ p(x_n, x) < \epsilon$ for all $n \geq n_0$. Hence, by Lemma 6 (iv) $p(x_n, x) \ll \epsilon c \ll c$ for all $n \geq n_0$. The proof of the converse of implication (ii) is similar. Now it is possible to say that the part (iii) of Lemma 11 is an immediate consequence of (i) and (ii).

Proposition 13. (See [7]) Let (X, p) is complete TVS-CMS and $T : X \rightarrow X$ satisfy the contractive condition

$$p(Tx, Ty) \leq kp(x, y) \quad (1)$$

for all $x, y \in X$ and $0 \leq k < 1$. Then, T has a unique fixed point in X . Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to fixed point.

Lemma 14. Let (X, d) be a cone metric space over a locally convex space (E, S) , where S is the family of seminorms defining the locally convex topology. Let $\{x_n\}$ be a sequence in E . Then

- (i) $x_n \rightarrow x$ in (X, d) if and only if $d(x_n, x) \rightarrow 0$ in (E, S) .
- (ii) $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0$ in (E, S) .

Proof. (i) Suppose that $\{x_n\}$ converges to x . Let $\varepsilon > 0$ and $p \in S$ be given. Choose $c \gg 0$ such that $p(c) < \varepsilon$. This is possible by taking $c = \frac{\varepsilon c_0}{2p(c_0)}$, where c_0 is an interior point of P . Then there is n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$. Then by normality of the cone P we have $p(d(x_n, x)) \leq p(c) < \varepsilon$ for all $n > n_0$. This means $d(x_n, x) \rightarrow 0$ in (E, S) . Conversely, suppose that $d(x_n, x) \rightarrow 0$ in (E, S) . For $c \in E$ with $c \gg 0$ find $\delta > 0$ and $p \in S$ such that $p(b) < \delta$ implies $b \ll c$. For this δ and this p find n_0 such that $p(d(x_n, x)) < \delta$ for all $n > n_0$ and so $d(x_n, x) \ll c$ for all $n > n_0$. Therefore $x_n \rightarrow x$ in (X, d) .

(ii) The proof is similar to that in (i). \square

Lemma 15. Let (X, d) be a TVS-cone metric space over a normal cone of a locally convex space (E, S) , where S is the family of seminorms defining the locally convex topology. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \rightarrow x$, $y_n \rightarrow y$. Then $d(x_n, y_n) \rightarrow d(x, y)$ in (E, S) .

Proof. Let $\varepsilon > 0$ and $p \in S$ be given. Choose $c \in E$ with $c \gg 0$ such that $p(c) < \frac{\varepsilon}{6}$. From $x_n \rightarrow x$ and $y_n \rightarrow y$, find n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$ and $d(y_n, y) \ll c$. Then for all $n > n_0$ we have

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) \leq d(x, y) + 2c,$$

and

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \leq d(x_n, y_n) + 2c.$$

Hence

$$0 \leq d(x, y) + 2c - d(x_n, y_n) \leq 4c$$

and so by the normality of P we obtain

$$p(d(x_n, y_n) - d(x, y)) \leq p(d(x, y) + 2c - d(x_n, y_n)) + p(2c) \leq 6p(c) < \varepsilon.$$

Therefore $d(x_n, y_n) \rightarrow d(x, y)$ in (E, S) . \square

2 Topological TVS-Cone Metric Spaces

Theorem 16. *Every TVS-CMS is a first countable topological space*

Proof. Let (X, ρ) be a TVS-CMS over a cone P with nonempty interior. Let $\beta_\rho = \{B(x, c) : x \in X, c \gg 0\}$. If we show that for all $c_1 \gg 0$, $c_2 \in E$ there exists a $c \gg 0$ such that $c \ll c_1$ and $c \ll c_2$ then we would be able to infer that $\tau_\rho = \{U \subseteq X : \forall x \in U \exists c \gg 0 \text{ with } B(x, c) \subseteq U\} \cup \{\Phi\}$ is a topology on X with basis β_ρ . To this end let $c_1 \gg 0$ and $c_2 \gg 0$ be given and hence find $\delta > 0$ and $q \in S$ such that $q(b) < \delta$ implies $b \ll c_2$. Also, choose n_0 such that $\frac{1}{n_0} < \frac{\delta}{q(c_1)}$. Let $c = \frac{c_1}{n_0}$, then $c \gg 0$, $c \ll c_1$ and $c \ll c_2$. For the same interpretation (X, ρ) is first countable. Actually, it can be shown that $\beta_x = \{B(x, \frac{c_0}{n}) : n \in \mathbb{N}\}$ is a countable local base at x for some $c_0 \gg 0$, where $B(x, \frac{c_0}{n}) = \{y \in X : \rho(x, y) \ll c_0\}$. \square

Corollary 17. *A mapping from a TVS-CMS to an arbitrary topological space is continuous if and only if it is sequentially continuous*

The proof follows by Theorem 16.

Corollary 18. *Every TVS-CMS (X, p) is topologically isomorphic to its correspondent metric space (X, d_p)*

The proof follows by Remark 12 and Corollary 17.

Corollary 19. *Every TVS-CMS is a paracompact topological space and hence T_4 space.*

The proof follows by Theorem 16, Remark 12 and that paracompactness is a topological property.

Corollary 19 above generalizes the result obtained recently by A. Sönmez in [15]. No normality assumption is assumed in our case.

Corollary 20. *Every TVS-CMS is topologically isomorphic to a dense subspace of a complete metric space.*

The proof follows by Corollary 18 and the completion theorem of usual metric spaces.

Note that in Corollary 20 above, we didn't say that the imbedding is isometrically isomorphic. This is because isometries are only defined between TVS cone metric spaces over the same E . However, for example, if (X, p) is a CMS over a normal cone P of a Banach E , then (X, p) is isometrically isomorphic to a dense subspace of a complete CMS over the same cone P (see [2]).

Corollary 21. *A TVS-CMS is compact if and only if it is sequentially compact.*

The proof follows by Corollary 18, compactness and sequential compactness are topological properties and that a subset of a metric space is compact if and only if it is sequentially compact.

Corollary 22. *Every TVS-CMS is of second category.*

The proof followed by Theorem 16 and that being of second category is a topological property.

Definition 23. *A subset A of a TVS-CMS (X, p) is called sequentially closed if whenever $x_n \in A$ with $x_n \rightarrow x$ then $x \in A$.*

Corollary 24. *A subset A of a TVS-CMS (X, p) is sequentially closed if and only if A is closed.*

The proof follows by Theorem 16.

Proposition 25. *Let (X, p) be a TVS-CMS. Then the ball $\overline{B}(x, c_0) = \{y \in X : p(x, y) \leq c_0\}$, $c_0 \gg 0$, is sequentially closed and hence closed.*

Proof. Let $y_n \in \overline{B}(x, c_0)$ be a sequence such that $y_n \rightarrow y$. Then, for each $m \in \mathbb{N}$ there exists k_m such that $p(x, y) \leq p(x, y_{k_m}) + p(y_{k_m}, y) \ll c_0 + \frac{c_0}{m}$. Hence, $\frac{c_0}{m} + c_0 - p(x, y) \in P$. Since P is closed then we conclude that $c_0 - p(x, y) \in P$ and so $p(y, x) \leq c_0$. \square

The nonlinear scalarization is a way to generate metric topologies comparable to the TVS-cone metric topologies. The following theorem shows how to generate comparable metric topologies by using the generating system of seminorms.

Theorem 26. *Let (X, p) be a TVS-CMS over a cone P in a metrizable locally convex topological vector space (E, S) , $S = \{q_k : k = 1, 2, \dots\}$ the set of seminorms generating the topology of E . Then*

(i) *the function $d_S : X \times X \rightarrow \mathbb{R}$ defined by*

$$d_S(x, y) = \inf\{h(u) : p(x, y) \leq u, u \in P\},$$

where $h(u) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_k(u)}{1+q_k(u)}$, is a metric on X .

(ii) *the metric topology τ_S induced by d_S is finer than the cone metric topology τ_p induced by p . In this case $d_S(x, y) = \inf\{h(u) : p(x, y) \leq u, u \in P\}$,*

(iii) *If E is a real Banach space (i.e (X, p) is a CMS), then (X, p) and (X, d_S) are topologically isomorphic.*

Proof. (i) Clearly $0 \leq d_S(x, y) < \infty$ and $d_S(x, y) = d_S(y, x)$ for all $x, y \in X$. If $d_S(x, y) = 0$ then there exists $\{u_n\} \in E$ such that $p(x, y) \leq u_n$ and $h(u_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence, $q_k(u_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $k = 1, 2, \dots$. Let $c \gg 0$ be given then find $\eta > 0$ and $k_0 \in \mathbb{N}$ such that $q_{k_0}(b) < \eta$ implies $b \ll c$. Since $q_{k_0}(u_n) \rightarrow 0$ as $n \rightarrow \infty$, find n_0 such that $q_{k_0}(u_n) < \eta$ for all $n \geq n_0$. Hence, $p(x, y) \leq u_n \ll c$. Therefore, $p(x, y) \in P \cap -P$ which implies that $p(x, y) = 0$ and so $x = y$. Now if $x = y$ then $p(x, y) = 0$, which implies that $d_S(x, y) \leq h(u)$ for all $u \geq 0$ and hence $0 \leq d_S(x, y) \leq h(0) = 0$ and so $d_S(x, y) = 0$.

To prove the triangle inequality, for $x, y, z \in X$ we have,

$$\forall \varepsilon > 0 \quad \exists u_1 : h(u_1) < d_S(x, z) + \varepsilon, \quad p(x, z) \leq u_1,$$

and

$$\forall \varepsilon > 0 \quad \exists u_2 : h(u_2) < d_S(z, y) + \varepsilon, \quad p(z, y) \leq u_2,$$

But $p(x, y) \leq p(x, z) + p(z, y) \leq u_1 + u_2$ implies that

$$d_S(x, y) \leq h(u_1 + u_2) \leq h(u_1) + h(u_2) \leq d_S(x, z) + d_S(z, y) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary then we have $d_S(x, y) \leq d_S(x, z) + d_S(z, y)$.

(ii) We show that if $x_n \rightarrow x$ in (X, d_S) then $x_n \rightarrow x$ in (X, p) . First from the definition of d_S we can find $\{u_n\}$ in E such that

$$h(u_n) < d_S(x_n, x) + \frac{1}{n}, \quad p(x_n, x) \leq u_n.$$

Now, since $h(u_n) \rightarrow 0$ as $n \rightarrow \infty$ then $u_n \rightarrow 0$ in (E, S) . If $c \gg 0$ is given, find n_0 such that $u_n \ll c$ for all $n \geq n_0$. This implies that $p(x_n, x) \ll c$ for all $n \geq n_0$. That is $x_n \rightarrow x$ in (X, p) .

(iii) Actually, this part is true even E is a seminormed space. Let $\varepsilon > 0$ be given and choose $c \in E$ such that $c \gg 0$ and $\|c\| < \varepsilon$. This is possible since $\text{int}(P)$ is nonempty. Namely, let $c = \frac{\varepsilon c_0}{2\|c_0\|}$. Now, find n_0 such that $p(x_n, x) \ll c$ for all $n \geq n_0$. Hence, $d_S(x_n, x) \leq \|c\| < \varepsilon$ for all $n \geq n_0$. \square

Note that to prove that (iii) above is true in a certain metrizable locally convex space, we need to show that for every $\varepsilon > 0$ there exists $c \gg 0$ such that $h(c) < \varepsilon$.

Example 27. Let $X = \omega$: the vector space of all real sequences, $E = \omega$ and $P = \{x = \{x_i\} : x_i \geq 0, \text{ for all } i\}$. Then ω is a complete metrizable locally convex space (Fréchet space) when its topology is generated by the seminorms $\{q_k : k = 1, 2, 3, \dots\}$ where $q_k(x) = \sum_{i=1}^k |x_i|$, $x = \{x_i\} \in \omega$. Clearly P is a normal cone in E but $\text{int}(P) = \Phi$ [14]. In this connection, it is known that a Fréchet space admits a continuous norm if and only if it does not contain ω as a complemented subspace. On X define $d(x, y) = \{\|x_i - y_i\|\}_i$, $x = \{x_i\}$, $y = \{y_i\} \in X$. Then (X, d) is a TVS-cone metric space over E . In E , we can define h naturally by

$$h(u) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|u_k|}{1 + |u_k|}, \quad u = \{u_k\} \in E.$$

It can be shown that for every $\varepsilon > 0$ there exists $c \gg 0$ such that $h(c) < \varepsilon$. Actually, for $\varepsilon > 0$, choose $c = (\varepsilon, 0, 0, \dots) \gg 0$, then $h(c) < \varepsilon$.

Definition 28. A subset A of a TVS-CMS (X, p) is called bounded above if there exists $c \in E$, $c \gg 0$ such that $p(x, y) \leq c$ for all $x, y \in A$, and is called bounded if $\delta(A) = \sup\{d(x, y) : x, y \in A\}$ exists in E . If the supremum does not exist, we say that A is unbounded.

Proposition 29. Let (X, p) be TVS-CMS over a strongly minihedral normal cone P in (E, S) and $A \subset X$. Then, A is bounded if and only if $\delta_q(A) = \sup\{q(p(x, y)) : x, y \in A\} < \infty$ for all $q \in S$ (i.e. $\{p(x, y) : x, y \in A\}$ is bounded in (E, S)).

Proof. Assume A is bounded and hence find $c \gg 0$ such that $p(x, y) \leq c$ for all $x, y \in A$. Let $q \in S$ be given, then by normality $\delta_q(A) = \sup\{q(p(x, y)) : x, y \in A\} \leq q(c) < \infty$.

Conversely, assume $\delta_q(A) < \infty$ for all $q \in S$ and fix some $c_1 \gg 0$. Then find $q_0 \in S$ and $\eta > 0$ such that $q_0(b) < \eta$ implies $b \ll c_1$. For each $x, y \in A$, let $c_{x,y} = \frac{\eta p(x,y)}{2q_0(p(x,y))}$. Then $q_0(c_{x,y}) = \frac{\eta}{2} < \eta$ implies $c_{x,y} \ll c_1$ and so $\frac{2q_0(p(x,y))}{\eta} c_1 - p(x, y) \in \text{int}(P)$. Therefore, $p(x, y) \ll \frac{2q_0(p(x,y))}{\eta} c_1 \leq \frac{2\delta_{q_0}}{\eta} c_1 = c$. Since P is strongly minihedral then A is bounded. \square

In fact, Corollary 29 above shows that if the cone is normal then A is bounded above if and only if $\{p(x, y) : x, y \in A\}$ is bounded in (E, S) .

3 Fixed Point Theory Remarks

Theorem 30. Let (X, p) be a TVS-CMS and (X, d_p) its correspondent metric space, $d_p = \xi_e \circ p$. Assume $T : X \rightarrow X$ is a mapping such that $p(Tx, Ty) \leq \varphi(p(x, y))$ for all $x, y \in X$, where $\varphi : P \rightarrow P$ is increasing and $\varphi(re) \leq r\varphi(e)$ for every real number $r \geq 0$. Then, there exists linear, increasing and continuous mapping $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $d_p(Tx, Ty) \leq \phi(d_p(x, y))$, for all $x, y \in X$.

Proof. Define $\phi(t) = \xi_e(\varphi(e))t$ for every $t \geq 0$. Then clearly, ϕ is linear, continuous, increasing, $\phi(0) = 0$ and $\phi(\xi_e(e)) = \phi(1) = \xi_e(\varphi(e))$. To prove our claim, let $\varepsilon > 0$ be given and find $r \in \mathbb{R}$ such that $0 \leq d_p(x, y) < r < d_p(x, y) + \varepsilon$, where $p(x, y) \in re - P$ (or $p(x, y) \leq re$). Now, by Lemma 6 and that φ is increasing, we have

$$d_p(Tx, Ty) = \xi_e(p(Tx, Ty)) \leq$$

$$\xi_e(\varphi(p(x, y))) \leq \xi_e(\varphi(re)) \leq r\xi_e(\varphi(e)) = r\phi(1) = \phi(r) \leq \phi(d_p(x, y)) + \phi(\varepsilon) \quad (2)$$

Since ε is arbitrary and $\phi(0) = 0$ we conclude that $d_p(Tx, Ty) \leq \phi(d_p(x, y))$. \square

Theorem 31. [16] Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a mapping satisfying

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad \text{for all } x, y \in X,$$

where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is uppersemicontinuous from the right, satisfying $\phi(t) < t$ for $t > 0$. Then, T has a unique fixed point, say u , and $T^n x \rightarrow u$ as $n \rightarrow \infty$ for all $x \in X$.

Proposition 32. Let (X, p) be a complete TVS-CMS and let $T : X \rightarrow X$ be a mapping satisfying

$$p(Tx, Ty) \leq \varphi(p(x, y)), \quad \text{for all } x, y \in X,$$

where $\varphi : P \rightarrow P$ is increasing, $\varphi(re) \leq r\varphi(e)$ for every real number $r \geq 0$, and $\xi_e(\varphi(e)) < 1$. Then, T has a unique fixed point, say u , and $T^n x \rightarrow u$ as $n \rightarrow \infty$ for all $x \in X$.

The proof follows by Theorem 31 and Theorem 30.

Theorem 33. Let (X, p) be a TVS-CMS and (X, d_p) its correspondent metric space, $d_p = \xi_e \circ p$. Assume $T : X \rightarrow X$ is a mapping such that $p(Tx, Ty) \leq \varphi(p(x, y))$ for all $x, y \in X$, where $\varphi : P \rightarrow P$ is increasing, continuous, $\varphi(0) = 0$ and $\varphi(re) \ll re$ for every real number $r > 0$. Then, there exists a continuous increasing mapping $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(t) < t$ for $t > 0$ and $d_p(Tx, Ty) \leq \phi(d_p(x, y))$, for all $x, y \in X$.

Proof. Define $\phi(r) = \xi_e(\varphi(re))$ for every $r \geq 0$. Then clearly, by Lemma 6, ϕ is continuous, increasing, $\phi(0) = 0$ and $\phi(r + s) \leq \phi(r) + \phi(s)$ for all $r, s > 0$. To prove our claim, let $\varepsilon > 0$ be given and find $r \in \mathbb{R}$ such that $0 \leq d_p(x, y) < r < d_p(x, y) + \varepsilon$, where $p(x, y) \in re - P$ (or $p(x, y) \leq re$). Now, by Lemma 6 and that φ is increasing, we have

$$d_p(Tx, Ty) = \xi_e(p(Tx, Ty)) \leq$$

$$\xi_e(\varphi(p(x, y))) \leq \xi_e(\varphi(re)) = \phi(r) \leq \phi(d_p(x, y)) + \phi(\varepsilon) \quad (3)$$

Since ε is arbitrary and $\phi(0) = 0$ we conclude that $d_p(Tx, Ty) \leq \phi(d_p(x, y))$. \square

Proposition 34. Let (X, p) be a complete TVS-CMS and let $T : X \rightarrow X$ be a mapping satisfying

$$p(Tx, Ty) \leq \varphi(p(x, y)), \quad \text{for all } x, y \in X,$$

where $\varphi : P \rightarrow P$ is increasing uppersemicontinuous from right, $\varphi(re) \ll re$ for every real number $r > 0$. Then, T has a unique fixed point, say u , and $T^n x \rightarrow u$ as $n \rightarrow \infty$ for all $x \in X$.

The proof follows by Theorem 31 and Theorem 33.

Remark 35. Theorem 30 above can also be used to generalize some quasi-contraction type fixed point theorems from metric spaces to TVS-cone metric spaces.

Remark 36. *Even it is possible to find metric spaces which are topologically isomorphic to the original TVS-CMS, it is still interesting to ask or comment on the following:*

(i) *Is it possible to give rise to nonlinear contraction type conditions with φ not necessarily increasing, by the nonlinear scalarization from TVS-CMS to usual metric spaces and thus provides trivial proofs to the TVS-CMS counterpart of the results such as in [16] and without any regularity assumption. If it is not the case, can we adapt the proof in [16] to regular complete TVS-CMS? That is can we remove the monotonicity of φ from Theorem 33.*

(ii) *There are many properties which are not topological thus they can not pass from (X, p) to (X, d_p) .*

(iii) *Proving a completion theorem for TVS-CMS is still interesting. Actually, even (X, p) and (X, d_p) are topologically isomorphic we can't say they are isometric. See the discussion in the previous section.*

(iv) *Is it trivial to generalize all Meir-Keeler type fixed point theorems to TVS-CMS and just by applying the nonlinear scalarization function?*

(v) *For a closed and convex subset C of a Banach space and $T : C \rightarrow C$ satisfies the conditions $0 \leq s + |a| - 2ab < 2(a + b) 4ad(Tx, Ty) + b(d(x, Tx) + d(y, Ty)) \leq sd(x, y)$, $d(x, y) = \|x - y\|$, T has at least one fixed point (for example see [17]). The nonlinear scalarization ξ_e can not be applied to generalize such kind of theorems to TVS-CMS. However, in the proof of such cases Lemma 15 will be crucial.*

(vi) *We may conclude that the nonlinear scalarization does not work if extra normality or regularity assumptions on the cone P are requested. For example, the following theorem which has been proved in [20]:*

Theorem 37. [20] *Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in \text{int}P$ for each $x, y \in P$ with $x \neq y$. Let $T : X \rightarrow X$ be a mapping satisfying the inequality*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \text{for } x, y \in X \quad (4)$$

where $\text{int}P \cup \{0\} \rightarrow \text{int}P \cup \{0\}$ is a continuous and monotone function with

(i) $\varphi(t) = 0$ if and only if $t = 0$

(ii) $\varphi(t) \ll t$ for $t \gg 0$,

(iii) either $\varphi(t) \leq d(x, y)$ or $d(x, y) \leq \varphi(t)$ for $t \in \text{int}P \cup \{0\}$ and $x, y \in X$.

Then T has a unique fixed point in X .

It is clear that the properties of the nonlinear scalarization are not enough to rise the condition (4) to usual metric space.

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